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DETERMINATION OF CRITICAL LOADS RESPONSIBLE FOR THE DEVELOPMENT OF INITIALLY WIDE CRACKS

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DETERMINATION OF CRITICAL LOADS RESPONSIBLE FOR THE DEVELOPMENT OF INITIALLY WIDE CRACKS[1]

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ABSTRACT. Discussion of a problem in the theory of brittle-fracture cracks, where an elastic body (infinite plane) is weakened by an initially wide crack. An analysis of the two-dimensional problem in the theory of elasticity for an infinite plane containing an incision of arbitrary shape, with one symmetry axis and one or two cuspidal points at the contour, is reduced to the solution of two systems of linear algebraic equations. It is shown that the width of the initial crack has only a slight effect on the value of the critical load required to initiate the development of a crack. This is seen to indicate that the representation of a crack as a cut of zero width is justified even for rather wide cracks.

The problem of the theory of cracks associated with brittle fracture is considered for the case when the elastic body (infinite plane) is weakened by an initially extended crack. The investigation of the two-dimensional problem of the theory of elasticity for an infinite plane weakened by an arbitrary notch with one axis of symmetry, with one or two cuspidal points at the contour, subjected to tension "at infinity" is reduced to the solution of two simple systems of linear algebraic equations. /63*

Two simple examples are investigated in detail following the procedures in ref. 1, 5-8. It is shown that the extension of the crack has an insignificant effect on the value of the critical load which is necessary to start the development of the crack. This makes it possible to conclude that the representation of the crack in the form of a notch with zero thickness is entirely justified even for sufficiently extended cracks.

Section 1. Let us consider the two-dimensional problem of the theory of elasticity for an infinite plane xOy weakened by a notch

$$x = R\left[\cos\varphi + \sum_{n=1}^{N} c_n \cos(kn-1)\varphi\right]; \; y = R\left[\sin\varphi - \sum_{n=1}^{N} c_n \sin(kn-1)\varphi\right], \qquad (1.1)$$

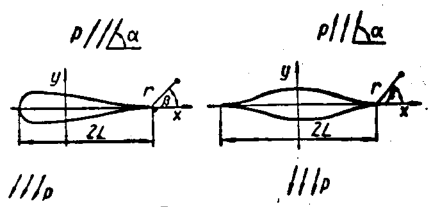
where R, c_n are real parameters, k=1, 2 is the number of cuspidal points at the contour; $0 \leq \varphi \leq 2\pi$.

The contour (1.1) has one (k=1) or two (k=2) cuspidal points at the x axis

* Numbers in the margin indicate pagination in the original foreign text.

[1] We have in mind cracks in which the initial distance between opposite borders may achieve a substantial value.

when a tensile force p acts at infinity, at an angle α with respect to the x axis (see figure).



The boundary conditions (ref. 4) in this case have the form

$$\varphi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)} \mp \overline{\psi(\sigma)} = 0;$$
$$\overline{\varphi(\sigma)} + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\varphi'(\sigma) + \psi(\sigma) = 0. \qquad (1.2)$$

The function $\omega(\zeta)$ transforms the region outside the contour (1.1) into the region inside a unit circle in the plane ζ and has the form

$$z = \omega(\zeta) = R\left(\zeta + \sum_{n=1}^{N} c_n \zeta^{1-kn}\right), \qquad (1.3)$$

Where the coefficients c_n satisfy condition (ref. 2)

$$\omega'(\zeta) = R(1 - \zeta^{-k}) Q_N(\zeta), \qquad (1.4)$$

where Q_N is a polynomial in negative powers of ζ, all of whose roots lie inside /64
a unit circle in the plane ζ.

As in refs. 2, 3, we assume that the function $\varphi(\zeta)$ has no singularities due to the cuspidal points, while the function $\psi(\zeta)$ has singularities in the form of simple poles at the points of unit circumference γ, corresponding to the cuspidal points at the contour (1.1).

Let us represent $\varphi(\zeta)$ by means of the following relationship:

$$\varphi(\zeta) = Rp\left(\frac{\zeta}{4} + \sum_{n=1}^{N} a_n \zeta^{1-kn}\right), \qquad (1.5)$$

where

$$a_n = \alpha_n + i\beta_n.$$

Following ref. 4, we write the function $\psi(\zeta)$ in the following form:

$$\psi(\zeta) = -\frac{Rp}{2} e^{-2i\alpha}\zeta + \sum_{n=0}^{\infty} \frac{b_n}{\zeta^n}. \qquad (1.6)$$

We rewrite the second boundary condition from relationships (1.2) in the form

$$-\omega'(\sigma)\,\psi(\sigma)=\omega'(\sigma)\,\overline{\varphi}\left(\frac{1}{\sigma}\right)+\omega\left(\frac{1}{\sigma}\right)\varphi'(\sigma). \tag{1.7}$$

Comparing the coefficients in front of the same powers of σ in the expansions of both sides of (1.7) we obtain two systems of linear algebraic equations

$$\alpha_p+\sum_{n=1}^{N-p}(1-kn)\,c_n\alpha_{n+p}+\sum_{n=1}^{N-p}(1-kn)\,\alpha_n c_{n+p}+\frac{c_p}{4}=A; \tag{1.8}$$

$$\beta_p+\sum_{n=1}^{N-p}(1-kn)\,c_n\beta_{n+p}-\sum_{n=1}^{N-p}(1-kn)\,\beta_n c_{n+p}=B,\ (p=1,2\ldots N). \tag{1.9}$$

In the first and second systems when k=1 and k=2 we have, respectively,

$$A=\begin{cases}0, & p\neq 2;\\ \frac{1}{2}\cos 2\alpha, & p=2;\end{cases}\qquad B=\begin{cases}0, & p\neq 2;\\ \frac{1}{2}\sin 2\alpha, & p=2;\end{cases}$$

$$A=\begin{cases}0, & p>1;\\ \frac{1}{2}\cos 2\alpha, & p=1;\end{cases}\qquad B=\begin{cases}0, & p>1;\\ \frac{1}{2}\sin 2\alpha, & p=1.\end{cases}$$

Multiplying both sides of (1.7) by $\frac{1}{2\pi i}\frac{1}{\sigma-\zeta}$ and integrating over γ we obtain the function $\psi(\zeta)$ in closed form

$$\psi(\zeta)=-\overline{\varphi}\left(\frac{1}{\zeta}\right)-\frac{\omega\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}\varphi'(\zeta). \tag{1.10}$$

Section 2. As an example we consider two simple contours with one or two cuspidal points whose equations for k=1 and k=2 can be expressed in a parametric form as follows: /65

$$x=R\left[(2-b)\cos\varphi+\frac{b}{2}\cos 2\varphi\right];\ y=R\left(b\sin\varphi-\frac{b}{2}\sin 2\varphi\right); \tag{2.1}$$

$$x=R\left[(2-b)\cos\varphi+\frac{b}{3}\cos 3\varphi\right];\ y=R\left(b\sin\varphi-\frac{b}{3}\sin 3\varphi\right). \tag{2.2}$$

where $0 \leqslant b < 1$.

Equations (2.1) and (2.2) represent the contours of the expanded cracks shown in the figure. By varying the parameter b, which characterizes the form of the notches (2.1) and (2.2), from 0 to unity we obtain a rectilinear notch for b=0 which gradually expands as b is increased and when b=1 the contour (2.1) transforms into a hypocycloid with 3 cuspidal points while the contour (2.2) transforms into an astroid.

The functions which produce a conformal transformation of the regions exterior to contours (2.1) and (2.2) into the interior of a unit circle are written in the form

$$\omega(\zeta) = R\left[\zeta + (1-b)\zeta^{-1} + \frac{b}{2}\zeta^{-2}\right], \quad (k=1); \tag{2.3}$$

$$\omega(\zeta) = R\left[\zeta + (1-b)\zeta^{-1} + \frac{b}{3}\zeta^{-3}\right], \quad (k=2). \tag{2.4}$$

Solving systems (1.8) and (1.9) for the considered cases we find expressions for the function $\Phi(\zeta) = \dfrac{\varphi'(\zeta)}{\omega'(\zeta)}$

$$\Phi(\zeta) = \frac{p}{4}\,\frac{\zeta^3 + (1-b-2e^{2i\alpha})\zeta + b}{(\zeta-1)(\zeta^2+\zeta+b)}, \quad (k=1); \tag{2.5}$$

$$\Phi(\zeta) = \frac{p}{4}\,\frac{(9-b^2)\zeta^4 - D\zeta^2 + b(9-b^2)}{(9-b^2)(\zeta^2-1)(\zeta^2+b)}, \quad (k=2). \tag{2.6}$$

Here

$$D = 6(3+b)\cos 2\alpha + 6i(3-b)\sin 2\alpha - (1-b)(3+b)^2.$$

Section 3. It follows from references 5 and 7 that the principal part of the stress tensor components in the neighborhood of the cuspidal points for the considered contours may be represented in the form

$$\sigma_r = \frac{1}{4\sqrt{2r}}\left[k_1\left(5\cos\frac{\beta}{2} - \cos\frac{3}{2}\beta\right) + k_2\left(-5\sin\frac{\beta}{2} + 3\sin\frac{3}{2}\beta\right)\right] + O(1);$$

$$\sigma_\beta = \frac{1}{4\sqrt{2r}}\left[k_1\left(3\cos\frac{\beta}{2} + \cos\frac{3}{2}\beta\right) - \right. \tag{3.1}$$

$$-3k_2\left(\sin\frac{\beta}{2}+\sin\frac{3}{2}\beta\right)\Bigg]+O(1);$$

$$\tau_{r\beta}=\frac{1}{4\sqrt{2r}}\left[k_1\left(\sin\frac{\beta}{2}+\sin\frac{3}{2}\beta\right)+k_2\left(\cos\frac{\beta}{2}+3\cos\frac{3}{2}\beta\right)\right]+O(1).$$

Here r, β are polar coordinates; k_1, k_2 are the stress intensity coefficients /66 which are determined from the relationship (ref. 6)

$$k_1\sqrt{\frac{2}{r}}\cos\frac{\beta}{2}-k_2\sqrt{\frac{2}{r}}\sin\frac{\beta}{2}+O(1)=4\,\mathrm{Re}\,\Phi(1+\zeta_1), \tag{3.2}$$

where ζ_1 assumes the following form for contours (2.1) and (2.2)

$$\zeta_1=\sqrt{\frac{r}{R\left(1+\frac{b}{2}\right)}}\,e^{\frac{i\beta}{2}},\quad (k=1); \tag{3.3}$$

$$\zeta_1=\sqrt{\frac{r}{R(1+b)}}\,e^{\frac{i\beta}{2}},\quad (k=2). \tag{3.4}$$

Substituting the functions (2.5) and (2.6) into (3.2) and taking into account expression (3.3) and (3.4) we obtain the coefficients of stress intensity for the extended cracks (2.1) and (2.2)

$$k_1=p(1-\cos 2\alpha)\sqrt{\frac{R}{2+b}},\quad k_2=p\sin 2\alpha\sqrt{\frac{R}{2+b}},\quad (k=1); \tag{3.5}$$

$$k_1=p\frac{(3-b^2)-3\cos 2\alpha}{3-b}\sqrt{\frac{R}{2(1+b)}},\quad k_2=p\frac{3\sin 2\alpha}{3+b}\sqrt{\frac{R}{2(1+b)}},\quad (k=2). \tag{3.6}$$

In the limiting cases when b=0 (rectilinear notch) and b=1 (hypocycloid) we find k_1 and k_2 from equations (3.5) and (3.6) which coincides with the results obtained earlier in (ref. 5-8). Following references 5 and 8 we shall assume that the initial propagation of cracks takes place from the apex of the crack along a line, in which the normal tensile stresses reach a maximum permissible value.

The value of the critical load $p=p_{cr}$ which is necessary for the crack to go into a state of mobile equilibrium is obtained from the relationships (refs. 5, 6)

$$\lim_{r\to 0} \sqrt{r}\,\sigma_\beta = \frac{K}{\pi}; \tag{3.7}$$

$$\lim_{r\to 0} \sqrt{r} \left(\frac{\partial \sigma_\beta}{\partial \beta}\right)_{\beta=\beta_*} = 0, \tag{3.8}$$

where K is the coupling modulus (ref. 1); β_* is the angle which determines the initial direction of crack propagation.

By using the approach analogous to that of reference 5 we obtain the following expression from (3.1), (3.7), (3.8)

$$p_{cr} = \frac{K}{\pi} \cdot \frac{\sqrt{2}}{\cos^2 \frac{\beta_*}{2} \left(\tilde{k}_1 \cos \frac{\beta_*}{2} - 3\tilde{k}_2 \sin \frac{\beta_*}{2} \right)}, \tag{3.9}$$

where

$$\tilde{k}_1 = \frac{k_1}{p};\ \tilde{k}_2 = \frac{k_2}{p}.$$

When $0 \ll \alpha \ll \pi/2$ it follows from condition (3.8) that /67

$$\beta_* = -2 \arcsin \sqrt{\frac{6n^2 + 1 - \sqrt{8n^2 + 1}}{2(9n^2 + 1)}}, \tag{3.10}$$

where $n=k_2/k_1$.

We can see from equation (3.10) that for an extended crack (2.1) the angle β_* does not depend on the parameter b and will be exactly the same as in the case when the crack has the form of a rectilinear notch.

Let us determine the critical load $p=p_{cr}$ for the case when the tensile forces are directed along the y axis.

In this case it follows from expression (3.10) $B_*=0$ and from (3.9) we find

$$p_{cr} = \frac{K}{\pi} \cdot \frac{\sqrt{2}}{\tilde{k}_1}. \tag{3.11}$$

From relationships (3.5), (3.6), (3.11) we find

$$p_{cr} = \frac{K}{\pi} \sqrt{\frac{2}{L}} \sqrt{1 - \frac{b^2}{4}}, \quad (k = 1); \qquad (3.12)$$

$$p_{cr} = \frac{K}{\pi} \sqrt{\frac{2}{L}} \cdot \frac{6 - 2b}{6 - b^2} \sqrt{\left(1 - \frac{b}{3}\right)(1 + b)}, \quad (k = 2), \qquad (3.13)$$

where L is the half-length of the crack (see figure); and when k=1, L=R(2-b); when k=2L=2/3R(3-b).

It follows from equations (3.12) and (3.13) that as the crack extends, i.e. as the parameter b increases in value from 0 to unity, the value of the critical load decreases insignif antly compared with Griffiths' load.

The maximum decrease in p_{cr} does not exceed 14 percent for cracks with contour (2.1) and 8 percent for cracks with contour (2.2).

REFERENCES

1. Barenblatt, G. I., The Mathematical Theory of Equilibrium Cracks Which Form During Brittle Fracture (Matematicheskaya teoriya ravnovesnykh treshchin, obrazuyushchikhsya khrupkom nazrushenii) Prikladnaya Mekhanika i Tekhnicheskaya Fizika (PMTF) No. 4., 1961.
2. Kaminskiy, A. A. On Critical Loads Which Initiate the Development of Cracks Near a Hole (O kriticheskikh nagruzkakh, vyzyvayushchikh nachalo razvitiya treshchin vozle otverstiya) MTT, No. 4., 1966.
3. Kaminskiy, A. A. On the Critical Loads for Regions Weakended by Holes with Cracks, Collected Works "Stress Concentration" (O kriticheskikh nagruzkakh dlya oblastey, oslablennykh otverstiyami s treshchinami. Sb. "Kontsentratsiya napryazheniy") Kiev, Izd-vo "Naukova Dumka", 1965.
4. Muskhelishvili, N. I. Some Basic Problems in the Mathematical Theory of Elasticity (Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti) Izd-vo AN SSSR, 1954.
5. Panasyuk, V. V. Berezhnitskiy, L. T., and Kovchik, S. Ye. On Propagation of an Arbitrarily Oriented Rectilinear Crack During the Tension of a Plate (O raspredelenii proizvol'no oriyentirovannoy pryamolineynoy treshchiny pri rastyazhenii plastiny) Prikladnaya Mekhanika, Vol. 1., No. 2., 1965.
6. Panasyuk, V. V. On the Failure of Brittle Bodies in a Two Dimensional State of Stress (O razrushenii khrupkikh tel pri ploskom napryazhennom sostoyanii) Prikladnaya Mekhanika, Vol 1., No. 9., 1965.
7. Si, Paris, Erdogan. Stress Concentration Factors at the Apex of the Crack During Two-Dimensional Tension and Flexure of Plates (Kontsentratsii napryazheniy u vershiny treshchiny pri ploskom rastyazhenii i

izgibe plastin) Tp. Amerikanskogo ob-va Inzh.-Mekhanikov, E, 29, No. 2, 1962.

8. Erdogan, Sikh. On the Development of Cracks in Plates Under the Action of Longitudinal and Transverse Forces (O razvitii treshchin v plastinakh pod deystviyem prodol'noy i poperechnoy nagruzok, SAE, E, 85, No. 4, Transactions of the American Society of Mechanical Engineers, 1963.

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